

A note on the rightmost particle in a Fleming-Viot process.

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Abstract We consider N nearest neighbor random walks on the positive integers with a drift towards the origin. When one walk reaches the origin, it jumps to the position of one of the other $N - 1$ walks, chosen uniformly at random. We show that this particle system is ergodic, and establish some exponential moments of the rightmost position, under the stationary measure.

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1 Introduction

There is recent interest in approximating the limiting law of irreducible Markov processes conditioned not to hit some (forbidden) state [11, 15]. This limiting law is not guaranteed to exist, but when it does it is called a quasi-stationary distribution (QSD). For QSD in the context of Birth and Death chains, we refer to [9], and the situation treated there is one in which there is a one parameter family of QSD.

QSD are neither well understood, nor easily amenable to simulation. One proposal made by Burdzy, Holyst, Ingberman, and March [7] (in a particular setting) is to consider N independent Markov processes except that when one reaches the forbidden state, it jumps to the state of one of the other processes, chosen uniformly at random. The natural conjecture is that the empirical measure, under the stationary measure, converges to the QSD as the number N of processes goes to infinity. It is also natural to conjecture that the selected QSD is *the minimal*, in terms of average time needed to reach the forbidden state.

In this note, we consider N random walks on \mathbb{N} , in continuous time, with a drift towards the origin. When one random walk reaches the origin, it jumps instantly to the position of one of the other $N - 1$ walks, chosen uniformly at random. We call Fleming-Viot the interacting random walks just described. Indeed, this dynamics has a genetic interpretation.

- Positions of the walks are evolving genetic traits of N individuals.
- The forbidden state (here 0) is a lethal trait (*the selection mechanism*).
- At the moment an individual dies, another one, chosen uniformly at random, branches (*the branching mechanism*). This keeps the population size constant.

We establish a Foster's criteria, which gives ergodicity, as well as a control of small exponential moments of the rightmost walk. To state our main result, let ξ_T denote the position of the N interacting walks at time T , and let $E[\cdot|\xi_0 = \xi]$ denote average with respect of the law of the process ξ_T with initial condition ξ .

Theorem 1.1 *There are positive constants $K, \alpha, \kappa, \delta_0, A, c_1, c_2, c_3$ such that for $N \in \mathbb{N}$, time $T = A \log(N)$, any $\delta < \delta_0$, and $\xi \in \mathbb{N}^N$, we have*

$$E \left[\exp(\delta \max(\xi_T)) | \xi_0 = \xi \right] - \exp(\delta \max(\xi)) < -c_1 \mathbb{I}_{\max(\xi) > K \log(N)} e^{\delta \max(\xi)} + c_2 \mathbb{I}_{\max(\xi) > K \log(N)} e^{-\kappa T} e^{\delta \max(\xi)} + c_3 e^{\delta \alpha \log(N)}. \quad (1.1)$$

As a consequence, for N large enough there is a unique invariant measure λ_N for Fleming-Viot. Integrating over λ_N , there are $\beta, C > 0$ such that for any N , and $\delta < \delta_0$

$$\int \exp(\delta \max(\xi)) d\lambda_N(\xi) \leq C \exp(\delta \beta \log(N)). \quad (1.2)$$

This first elementary step is an important ingredient in the proof of the conjecture we alluded to above [14]. Also, it might be of independent interest in view of recent deep and comprehensive studies on the rightmost position in branching random walks [8, 1, 10, 4, 3, 5, 2, 12]. This selection of recent works is far from being exhaustive, but already shows the vitality of this issue.

The rest of paper is organized as follows. In Section 2 we define the model, and recall well-known large deviations estimates. In Section 3, we explain how to divide walks into groups with little correlations over a well chosen time period. In Section 4, we estimate the probability that the maximum displacement does not decrease. Finally, in Section 5, we establish Foster's criteria.

2 Model and Preliminaries

Here, we deal with continuous-time nearest neighbor random walks on \mathbb{N} , with rate p to jump right, and rate $q = 1 - p > p$ to jump left. The drift is $-v$ with $v = q - p > 0$. A single walk makes N_t jumps in the time period $[0, t]$, and its increments are denoted X_1, \dots, X_n , with $E[X_i] = -v$, and $\bar{X}_i = X_i + v$ denotes the centered variable. Note that

$$P\left(\sum_{i=1}^T (X_i + v) \geq xT\right) \leq \exp(-TI(x)), \quad (2.1)$$

with

$$I(x) = \sup_{\lambda > 0} \{\lambda x - \Lambda(\lambda)\} \quad \text{with} \quad \Lambda(\lambda) = \log(pe^\lambda + qe^{-\lambda}) + \lambda v. \quad (2.2)$$

Due to the nearest neighbor jumps of our walk, we have

$$I(v+1) = \log\left(\frac{1}{1-q}\right) \quad \text{and for} \quad x > v+1, \quad I(x) = \infty. \quad (2.3)$$

We define also $x \mapsto \tilde{I}(x) = 1 - \exp(-I(x))$, which is discontinuous at $v+1$ with

$$\tilde{I}(v) < \tilde{I}(v+1) = q \quad \text{and for} \quad x > v+1, \quad \tilde{I}(x) = 1. \quad (2.4)$$

Note that if N_T is Poisson of mean T , then

$$P\left(\sum_{i \leq N_T} (X_i + v) \geq xN_T\right) \leq \exp(-T\tilde{I}(x)). \quad (2.5)$$

On Poisson tails. We need two rough tail estimates on the Poisson clocks. Both are obvious and well-known. Assume χ, T are positive.

$$P(N_T \geq eT + \chi) \leq \exp(-T - \chi), \quad (2.6)$$

and

$$P(N_T \leq \frac{1}{e}T - \chi) \leq \exp(-(1 - 2/e)T - \chi), \quad (2.7)$$

Both are obtained readily by Chebychev's inequality. Indeed, we obtain (2.6) from

$$P(N_T \geq eT + \chi) \leq e^{-eT - \chi} E[e^{N_T}] = \exp(-T - \chi), \quad (2.8)$$

and we obtain (2.7) from

$$P(N_T \leq \frac{1}{e}T - \chi) \leq e^{T/e - \chi} E[e^{-N_T}] = \exp(-(1 - 2/e)T - \chi). \quad (2.9)$$

3 Independence

On the multitype branching of [6]. A key idea introduced in [6] is to embed the Fleming-Viot process into a multitype branching process whose space displacements and branching mechanism are independent, and which is *attractive*. We refer to Section 3 of [6] for a description of the multitype branching process, and recall here its main features. Assume that we start with N interacting random walks. This defines N types with which we associate N independent exponential clocks of intensity q with marks. The time realizations of the clock of type i have marks in the set of labels $\{1, \dots, N\} \setminus \{i\}$, and each mark is chosen uniformly at random from the $N - 1$ symbols. When clock i rings, and when its mark is j , each walk of type j branches into two children: one of type i and one of type j . The two children behave as independent random walks starting at the position of their parent. If \mathbb{D}_T denotes the population of individuals alive at time T , and $|\mathbb{D}_T|$ denotes its cardinal, it is easy to see the equality $E[|\mathbb{D}_T|] = |\mathbb{D}_0| \exp(qT)$. For an individual v alive at time T , we denote by $t \mapsto S_v(t)$ its trajectory for $t \in [0, T]$.

Independent groups of walks. A drawback of the multitype branching process is an exponentially growing population. Since, we use a time of order $\log(N)$, we cannot use here such an embedding. Even though in the Fleming-Viot process, all particles interact with each other, a simple observation is that as long as a particle has not touched the origin its trajectory is independent from the other ones, even though this trajectory might influence others.

To create some independence between walks, we decompose the interacting walks in two sets at time 0. We first fix a time T and a length L to be chosen later.

- The *blacks*, whose initial position is below L .
- The *reds*, whose initial position is above L .

Then, color changes as follows: if a black walk jumps on a red walk, it becomes instantly red. We interpret this jump as a *red binary branching*. Now, red walks are not independent from black walks because they might touch the origin before time T , and jump onto a black position. However, if $vT \ll L$, we expect this to be rare. To obtain independence, we add another color to our description: each red walk is coupled with a green walk which behaves identically in terms of move or branching

but with green children, except that when a green walk reaches the origin it continues its drifted motion on \mathbb{Z} (without selection mechanism). Thus, green walks behaves like independent random walks with branching at the times a black particle hit zero and chooses the label of a green walk. If R_0 is the first time one of the red walks touches the origin, we have that at time T , on the event $\{R_0 > T\}$, red and green positions are identical. The point of introducing green walks is that their branching times is independent of their positions. We denote with D_T^r, D_T^g, D_T^b the respective number of red, green and black walks at time T . Also $D_T = D_T^r \cup D_T^g \cup D_T^b = \{1, \dots, N\}$, and we still denote by $t \mapsto S_v(t)$, the trajectory of $v \in D_T$.

When embedding a group of walks into a branching process, we denote with $\mathbb{D}_T^r, \mathbb{D}_T^g, \mathbb{D}_T^b$ the respective number of red, green and black individuals in the multitype branching processes.

The key idea here is to work on a time of order $\log(N)$, to control the black walks by a multitype branching process, but to let the red walks (or rather the green walks) grow as in Fleming-Viot with a population bounded by N , and with branching due to independent black walks.

On the choice of time T and length L . We choose T large enough so that $q + \log(N)/T < 1$. We actually need a little more.

$$\kappa = \min \left(1 - 2/e, \tilde{I}\left(\frac{v}{2}\right) - \frac{\log(N)}{T}, 1 - q - \frac{\log(N)}{T} \right) > 0. \quad (3.1)$$

Once $T = A \log(N)$ satisfies (3.1), we set $L = eT$.

4 When things go wrong

We wish to estimate the probability of the event where the maximum displacement does not decrease. We thus define a *bad set* $B(T, L)$ as containing the following events:

- One red walk reaches the origin before time T (i.e. $\{R_0 \leq T\}$).
- One black walk travels a distance L upwards in a period $[0, T]$.
- The maximum displacement of a green walk in a time T is above $-\frac{v}{2e}T$.

Thus, on the complement on $B(T, L)$, green and red are identical, and

$$\max_{v \in D_T} S_v(T) - \max_{v \in D_0} S_v(0) \leq \max_{v \in D_T} \left(S_v(T) - S_v(0) \right) = \max_{v \in D_T^g} \left(S_v(T) - S_v(0) \right) < -\frac{v}{2e}T,$$

which implies that if $M(T) = \max_{v \in D_T} S_v(T)$

$$E \left[\mathbb{1}_{B^c(T, L)} \exp \left(\delta(M(T) - M(0)) \right) \middle| \xi(0) = \xi \right] \leq \exp \left(-\frac{v\delta}{2e}T \right). \quad (4.1)$$

We estimate next the probability of each event making up $B(T, L)$, with the following outcome.

Lemma 4.1 *For any $\xi \in \mathbb{N}^N$, we have, with κ as in (3.1),*

$$P(B(T, L) \middle| \xi(0) = \xi) \leq 4 \exp(-\kappa T). \quad (4.2)$$

4.1 A red walk does reach 0

Recall that $L = eT$. We embed the Fleming-Viot into a branching multitype, while keeping the red coloring. We need to estimate the probability that one red displacement gets below L units in a time period $[0, T]$. Note that to realize $\{R_0 < T\}$, there is $v \in \mathbb{D}_T^r$ such that the number of its time jumps N_T must be larger than L , and this is what we use.

$$\begin{aligned} P(R_0 < T | \xi(0) = \xi) &\leq E[|\mathbb{D}_T^r|] \times P(\exists t \leq T, \sum_{i \leq N_t} X_i < -eT) \\ &\leq E[|\mathbb{D}_T^r|] \times P(N_T > eT) \leq Ne^{qT} e^{-T} \leq e^{-\kappa T}. \end{aligned} \quad (4.3)$$

4.2 A large black displacement

Recall that at the time a black reaches 0, and jumps on a red walk, it ceases to be black to become red. We bound here the black walks with a multitype branching, assuming that blacks do only jump on blacks, with the effect that we are overestimating the black population. The estimates are similar to these of Section 4.1. We use that to make L steps right, a black walk must make L time-marks ($N_T > L$), and this event is estimated in (4.3).

4.3 Green's maximum too high

The key point is that the green branching times are independent of positions of the green. They depend only on the history of black walks. Also, the population of green walks is bounded by N . Thus, it is crucial here not to use the multitype branching of [6]: we estimate the probability that $\{\max_{v \in D_T^g} (S_v(T) - S_v(0)) > -vT/(2e)\}$. Define $\mathcal{N}_T(\gamma)$ as the number of green walks whose displacement during time period $[0, T]$ is larger than γ . Then,

$$E[\mathcal{N}_T(\gamma) | \xi(0) = \xi] = E[|D_T^g| | \xi(0) = \xi] \times P\left(\sum_{i \leq N_T} X_i > \gamma\right). \quad (4.4)$$

The reason is the independence of the branching times and displacements of the green walks. For $v \in D_T^g$, assume there are ν branchings before time T , say at times T_1, \dots, T_ν , and we have (for $X_i^{(k)}$ i.i.d. independent from $\{T_i, i \in \mathbb{N}\}$)

$$S_v(T) - S_v(0) = \sum_{i \in N[0, T_1]} X_i^{(1)} + \dots + \sum_{i \in N[T_\nu, T]} X_i^{(\nu)}. \quad (4.5)$$

As one conditions first on the black history up to time T , one fixes the times T_1, \dots, T_ν , and obtain that $N[0, T_1] + \dots + N[T_\nu, T]$ sums up to a Poisson variable $N[0, T]$ of intensity T , and most importantly

$$\sum_{i \in N[0, T_1]} X_i^{(1)} + \dots + \sum_{i \in N[T_\nu, T]} X_i^{(\nu)} = \sum_{i \in N[0, T]} X_i, \quad (4.6)$$

where the $\{X_i, i \in \mathbb{N}\}$ are i.i.d. increments independent of $N[0, T]$. We obtain, with κ defined in (3.1),

$$\begin{aligned}
P\left(\max_{v \in D_T^g} (S_v(T) - S_v(0)) > -\frac{v}{2e}T \mid \xi(0) = \xi\right) &\leq E\left[\mathcal{N}_T\left(-\frac{vT}{2e}\right) \mid \xi(0) = \xi\right] \\
&\leq NP\left(\sum_{i \leq N_T} \bar{X}_i > vN_T - \frac{v}{2e}T\right) \\
&\leq N\left(P\left(\sum_{i \leq N_T} \bar{X}_i > \frac{v}{2}N_T\right) + P\left(N_T < \frac{1}{e}T\right)\right) \\
&\leq N\left(\exp\left(-T\tilde{I}\left(\frac{v}{2}\right)\right) + \exp\left(-(1-2/e)T\right)\right) \leq 2\exp(-\kappa T).
\end{aligned} \tag{4.7}$$

5 Foster's criteria

We start with an estimate on the tail, and of the exponential moments.

5.1 On exponential moments

We deal here with the multitype branching process. Recall that $\mathbb{D}_0 = \{1, \dots, N\}$, and let $S(0) = \{S_v(0), v \in \mathbb{D}_0\}$.

Lemma 5.1 *For any T satisfying (3.1), and any $\chi > 0$*

$$P\left(\max_{v \in \mathbb{D}_T} (S_v(T) - S_v(0)) > eT + \chi \mid S(0) = \xi\right) \leq \exp(-\chi). \tag{5.1}$$

Proof. Since the branching mechanism is independent of positions

$$P\left(\max_{v \in \mathbb{D}_T} (S_v(T) - S_v(0)) > eT + \chi \mid S(0) = \xi\right) \leq E[|\mathbb{D}_T|] \times P\left(\sum_{i=1}^{N_T} X_i > eT + \chi\right). \tag{5.2}$$

Now, if \bar{X}_i denotes the centered variable, note that since the walk is nearest neighbor

$$P\left(\sum_{i=1}^{N_T} \bar{X}_i > (v+1)N_T\right) = 0.$$

Now,

$$\begin{aligned}
P\left(\sum_{i=1}^{N_T} X_i > eT + \chi\right) &= P\left(\sum_{i=1}^{N_T} \bar{X}_i > vN_T + eT + \chi\right) \\
&\leq P\left(\sum_{i=1}^{N_T} \bar{X}_i > (v+1)N_T\right) + P(N_T > eT + \chi) \\
&\leq 0 + P(N_T > eT + \chi)
\end{aligned} \tag{5.3}$$

Now, N_T is a Poisson variable of mean T , the standard estimate (2.6) leads to

$$P(N_T > eT + \chi) \leq \exp(-T - \chi). \tag{5.4}$$

Also, we have $E[|\mathbb{D}_T|] \leq N \exp(qT)$, and (5.2) and the choice of T in (3.1) yield

$$P\left(\max_{v \in \mathbb{D}_T} (S_v(T) - S_v(0)) > eT + \chi \mid S(0) = \xi\right) \leq N e^{qT} e^{-T-\chi} \leq e^{-\chi}. \quad (5.5)$$

■

We can state our main estimate.

Lemma 5.2 *Assume that $(1 - q)T > \log(N)$, and $\delta < 1$. Then, we have*

$$E \left[\exp \left(\delta \left(\max_{i \leq N} \xi_i(T) - \max_{i \leq N} \xi_i(0) \right) \right) \mid \xi(0) = \xi \right] \leq \frac{1}{1 - \delta} e^{\delta e T}. \quad (5.6)$$

Proof. For any random variable X , we have

$$\begin{aligned} E \left[e^{\delta X} \right] &= 1 + \int_0^\infty \delta e^{\delta u} P(X > u) du \\ &\leq 1 + \int_0^{eT} \delta e^{\delta u} du + \int_{eT}^\infty \delta e^{\delta u} P(X > u) du \\ &\leq e^{\delta e T} \left(1 + \int_0^\infty \delta e^{\delta u} P(X > u + eT) du \right). \end{aligned} \quad (5.7)$$

Now, using the tail estimate (5.1), we have

$$E \left[\exp \left(\delta \max_{v \in \mathbb{D}_T} (S_v(T) - S_v(0)) \right) \right] \leq e^{\delta e T} \left(1 + \int_0^\infty \delta e^{\delta u} e^{-u} du \right) \leq \frac{e^{\delta e T}}{1 - \delta}. \quad (5.8)$$

We now use the following bound to conclude

$$\begin{aligned} E \left[\exp \left(\delta \left(\max_{i \leq N} \xi_i(T) - \max_{i \leq N} \xi_i(0) \right) \right) \mid \xi(0) = \xi \right] &\leq E \left[\exp \left(\delta \left(\max_{v \in \mathbb{D}_T} S_v(T) - \max_{v \in \mathbb{D}_0} S_v(0) \right) \right) \mid S(0) = \xi \right] \\ &\leq E \left[\exp \left(\delta \max_{v \in \mathbb{D}_T} (S_v(T) - S_v(0)) \right) \mid S(0) = \xi \right]. \end{aligned} \quad (5.9)$$

■

5.2 Proof of Theorem 1.1

We recall the general strategy of the proof of Proposition 1.2 of [6] (The Foster criteria). We have a bad set $B(T, L)$ (which depends on T and L) which contains the cases where the maximum increases over a period $[0, T]$, or when black walks win over or influence red ones. First, there is a set K on which we do not expect the maximum to decrease, with

$$K = \left\{ \max_v (S_v(0)) < 3L \right\}. \quad (5.10)$$

Then, there is a good set where the maximum decreases:

$$K^c \cap B^c(T, L) \subset G = \left\{ \max_{v \in D_T} (S_v(T)) - \max_{v \in D_T} (S_v(0)) \leq -\frac{v}{2e} T \right\}. \quad (5.11)$$

Now, set $M_t = \max S_v(t)$. When ξ is the initial configuration, and when we work with $\xi \in K^c$, we have using Cauchy-Schwarz

$$\begin{aligned} \mathbb{1}_{\xi \in K^c} \left(E \left[e^{\delta M_T} \mid \xi(0) = \xi \right] - e^{\delta M_0} \right) &= \mathbb{1}_{\xi \in K^c} e^{\delta M_0} \left(E \left[e^{\delta(M_T - M_0)} (\mathbb{1}_{B(T,L)} + \mathbb{1}_G) \mid \xi(0) = \xi \right] - 1 \right) \\ &\leq \mathbb{1}_{\xi \in K^c} e^{\delta M_0} \left(P(B(T,L) \mid \xi(0) = \xi) E \left[\exp(2\delta(M_T - M_0)) \mid \xi(0) = \xi \right] \right)^{1/2} \\ &\quad - \mathbb{1}_{\xi \in K^c} e^{\delta M_0} \left(1 - e^{-\delta v T/2} \right). \end{aligned} \quad (5.12)$$

We know from Lemma 5.2 that for $\delta < 1/2$

$$E \left[\exp(2\delta(M_T - M_0)) \mid \xi(0) = \xi \right] \leq \frac{e^{2\delta e T}}{1 - 2\delta}.$$

Note also that on the set K , if we use Lemma 5.2

$$\mathbb{1}_K \left(E \left[e^{\delta M_T} \mid \xi(0) = \xi \right] - e^{\delta M_0} \right) \leq \mathbb{1}_K \exp(3\delta L + \delta e T). \quad (5.13)$$

Thus, adding (5.12) and (5.13), we obtain

$$E \left[e^{\delta M_T} \right] - e^{\delta M_0} \leq \mathbb{1}_K e^{3\delta L + \delta e T} - \mathbb{1}_{K^c} \left(1 - e^{-\delta v T/2} \right) e^{\delta M_0} + \mathbb{1}_{K^c} \left(P(B(T,L) \mid \xi(0) = \xi) \frac{e^{2\delta e T}}{1 - 2\delta} \right)^{1/2} e^{\delta M_0}. \quad (5.14)$$

We use now Lemma 4.1, and choose δ small enough so that $\kappa > 4\delta e$ with the result

$$E \left[e^{\delta M_T} \right] - e^{\delta M_0} \leq e^{3\delta L + \delta e T} - \mathbb{1}_{K^c} \left(1 - e^{-\delta v T/2} \right) e^{\delta M_0} + \mathbb{1}_{K^c} \frac{e^{-\kappa T/4}}{\sqrt{1 - 2\delta}} e^{\delta M_0}. \quad (5.15)$$

Inequality (5.15) is a Foster's criteria (see [13, Theorems 8.6 and 8.13]). This implies the first part of Theorem 1.1.

Now, as we integrate (5.15) with respect to the invariant measure, the left hand side of (5.14) vanishes, and we obtain

$$\left(1 - e^{-\delta v T/2} \right) \int_{K^c} e^{\delta M(\xi)} d\lambda^N(\xi) \leq \exp(3\delta L + \delta e T) + \exp\left(-\frac{\kappa}{4}T\right) \int_{K^c} e^{\delta M(\xi)} d\lambda^N(\xi). \quad (5.16)$$

With A large enough so that (3.1) holds with $T = A \log(N)$ and $L = eT$, the second part of Theorem 1.1 follows at once.

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